Written Comprehensive Examination-Theory

Department of Statistics, UC Irvine
Friday, June 17, 2016, 9:00 am to 1:00 pm

• There are 7 questions on the examination. Select any 5 of them to solve. If you attempt to solve more than 5 questions, you are only to turn in the 5 you want graded. If you turn in partial solutions to more than 5 questions, only 5 will be graded.

• Each of the 5 problems you attempt to solve will be worth equal credit, with each accounting for 20% of your final score on this examination.

• Your solutions to each problem should be written on separate sheets of paper. Label each sheet with your student identification number, the problem number, and the page number of that solution written in the upper right hand corner. For example, the labeling on a page may be:

ID# 912346378
Problem 2, page 3

• You have 4 hours to complete your solution. Please be prepared to turn in your exam at l:00 pm.

THEORY, Problem 1

Consider a typical water dispenser in our building. Let Y denote the supply (in gallons) at the beginning of a day, let X denote the amount (in gallons) dispensed during a day. Suppose that the joint pdf is

$$f(x,y) = \begin{cases} \frac{1}{2y} & \text{for } 0 < x < y \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) [10 pt] Are X and Y independent from each other? Why or why not?

(b) [10 pt] Show that the marginal distribution of Y is $Uniform(0,2)$.

(c) [20 pt] Show that the conditional distribution of X given $Y = y$ is $Uniform(0,y)$.

(d) [20 pt] Use the law of iterated expectations, i.e., marginal expectation equals expectation of conditional expectation, to verify that $E[X] = 1/2$. Use a similar trick to calculate $Var[X]$.

(e) [40 pt] There are 10 water dispensers in our building. We assume that the dispensers are independent. Within a given day, some dispensers work and the others are broken. It is known that the number of functional dispensers within a day follows $Binomial(10, 0.8)$.

   i. Verify that the mean of the total amount of water dispensed in a day is 4 gallons.

   ii. Find the variance of the total amount of water dispensed in a day.

THEORY, Problem 2

I have two remote controls, one is for my TV and the other is for my air conditioner (AC). Each remote control uses one battery. Once a battery breaks down, I replace it with a new one immediately. Let $N_{TV}(t)$ and $N_{AC}(t)$ denote the numbers of battery replacements needed within t years for the TV remote control and the AC remote control, respectively. Suppose that $N_{TV}(t)$ is a Poisson process at the rate of 2 replacements per year, $N_{AC}(t)$ is a Poisson process at the rate of 1 replacement per year. In addition, assume that the two Poisson processes are independent.

(a) [20 pt] Find $Pr(N_{TV}(1) = 1, N_{TV}(3) = 3, N_{AC}(3) = 2)$.

(b) [20 pt] Let $N(t) = N_{TV}(t) + N_{AC}(t)$. Show that $N(t) \sim Poisson(3t)$. Hint: you may use, without needing to prove, the following fact. If $Y \sim Poisson(\lambda)$, then $E[e^{sY}] = exp(\lambda(e^s - 1))$.

(c) [20 pt] I replaced exactly one battery in the first year. Find the probability that I replaced the battery of my TV remote control. Justify your answer.

(d) [20 pt] Let X_{TV} and X_{AC} denote the lifetime of a TV battery and an AC battery, respectively. Prove that X_{TV} follows an exponential distribution with mean lifetime 1/2 year.

(e) [20 pt] Let X be the minimum of X_{TV} and X_{AC}, i.e., X is the waiting time for a replacement that takes replace in either of the two remote controls. What is the distribution of X? Justify.

THEORY, Problem 3

Let $X_1, \cdots, X_n$ be a random sample from a uniform distribution on (0,1).

(a) [20 pt] Use a law of large numbers you learned in STAT200A to justify that the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ converges to a constant. Find the constant and describe the mode of convergence. You don't need to prove the law but you need to state it clearly.

(b) [20 pt] What is the limiting distribution of $\sqrt{n}(\bar{X}_n - 0.5)$? Justify.

(c) [20 pt] Find the limiting distribution of $-\sqrt{n}\,[log(\bar{X}_n) - log(0.5)]$. Justify your answer.

(d) [20 pt] Let $Y_i = -\frac{log(X_i)}{\theta}$, where θ be a positive number. Show that $Y_i, i = 1, \cdots, n$ are i.i.d random variables with pdf $f(y) = \theta e^{-\theta y}, y > 0$.

(e) [20 pt] Suppose that someone drew a random sample from $Uniform(0, 1)$ and then conducted the transformation described in (d). You didn't have access to the individual observations but you were given the sample size n and the sample mean $\bar{Y}$, where $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n}[-\frac{log(X_i)}{\theta}]$. Use asymptotic theories to construct an approximate 95% confidence interval for θ. Justify all the steps.

THEORY PROBLEM 4. Let X_1 and X_2 be independent random variables disrtibuted, respectively, as $N(\mu, \sigma^2)$ and $N(\mu, 4\sigma^2)$. Here, the mean parameter $\mu \in \mathbb{R}$ is unknown while the variance parameter σ^2 is known.

1. (5 points). A natural estimator for μ is $T_1(X_1, X_2) = \frac{1}{2}X_1 + \frac{1}{2}X_2$. Derive the bias, variance and mean-squared error of $T_1(X_1, X_2)$.

2. (10 points). Use the factorization theorem to show that $U(X_1, X_2) = 4X_1 + X_2$ is a sufficient statistic for μ. Then develop an estimator, denoted as $T_2(X_1, X_2)$ based on $U(X_1, X_2)$, that is unbiased for μ.

3. (5 points). Consider a third estimator $T_3(X_1, X_2)$ that takes the form

$$T_3(X_1, X_2) \;=\; \widetilde{a}_1 X_1 \;+\; \widetilde{a}_2 X_2$$

where $(\widetilde{a}_1, \widetilde{a}_2)$ are the weights that satisfy the following conditions: (i.) $(\widetilde{a}_1, \widetilde{a}_2) \in [0,1] \times [0,1]$; (ii.) $\widetilde{a}_1 + \widetilde{a}_2 = 1$; and $(\widetilde{a}_1, \widetilde{a}_2)$ minimize the mean-squared error, i.e.,

$$(\widetilde{a}_1, \widetilde{a}_2) \;=\; \arg \min_{(a_1, a_2)} \mathbb{E}\left[(a_1 X_1 + a_2 X_2) - \mu\right]^2.$$

4. (5 points). Based on a well-defined criterion rank these estimators (from best to worst).

5. (10 points). Using pivotal quantities based on each estimator $T_k(X_1, X_2)$ $(k = 1, 2, 3)$, derive exact 95% confidence intervals for μ. Which of these intervals would you choose? Give a brief explanation for your choice.

6. (15 points). Suppose that the uncertainty about the parameter μ is expressed via the prior distribution $\mu \sim N(\theta, \tau^2)$. (i.) Derive the Bayesian estimator for μ (based on the squared error loss function). (ii.) Derive a 95% Bayesian credible interval for μ.

7. (10 points). Suppose that you have some prior information about μ which is expressed in terms of the probability mass function $\mathbb{P}(\mu = \mu_0) = \pi_0$ and $\mathbb{P}(\mu = \mu_1) = \pi_1$ where $\pi_0 + \pi_1 = 1$. Thus, under this scenario, the prior belief is that there are only two possible values of μ which belong to the set $\{\mu_0, \mu_1\}$. Develop a Bayesian estimator for μ based on the observed data (X_1, X_2) (again, based on the squared error loss function).

8. (40 points). Derive the likelihood ratio test for $H_0 : \mu = \mu_0$ vs. $H_1 : \mu = \mu_1$.

THEORY PROBLEM 5. Note that this problem has two parts: Part A and Part B. The total score for Problem 5 is 100 which is split evenly for Parts A and B.

Problem 5 Part A. Let $\mathbf{X} = (X_1, X_2, \dots X_n)$ be a random sample from a Poison distribution with mean $\lambda \in (0, \infty)$. Here, you may assume that $n \geq 2$. Define $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$.

1. (10 points). For any constant $B \neq 0$, show that $\mathbb{E}B^T = \exp[n\lambda(B-1)]$.
2. (10 points). Derive a uniformly minimum variance unbiased estimator (UMVUE) for $g(\lambda) = \exp(-\lambda)$. Denote this to be $U(\mathbf{X})$.
3. (10 points). (i.) Derive the Cramér-Rao lower bound (CRLB) of the variance of all unbiased estimators for $g(\lambda)$. (ii.) Does the variance of the UMVUE $U(\mathbf{X})$ hit the CRLB? Explain or derive. (iii.) If not, it is possible to find an unbiased estimator for $g(\lambda)$ whose variance is exactly equal to the CRLB? (iv.) Since $U(\mathbf{X})$ is the UMVUE for $g(\lambda)$, is it possible to find *any* estimator for $g(\lambda)$ whose mean-squared error (MSE) is lower than that of the MSE of $U(\mathbf{X})$? Explain.
4. (20 points). Derive an approximate 95% confidence interval for the estimand $g(\lambda)$ (based on the MLE of λ, denoted $\widehat{\lambda}_n$) so that the interval falls entirely within the space of the estimand.

Problem 5 - Part B. Let $\mathbf{X} = (X_1, X_2, \dots X_n)$ be a random sample from a normal distribution with unknown mean $\mu \in \mathbb{R}$ and known variance of $\sigma^2 = 1$. Denote the cumulative distribution of the standard normal to be $\Phi(\cdot)$. The goal in this problem is to estimate $g(\mu) = \mathbb{P}(X_1 < c) = \Phi(c - \mu)$ where c is some known fixed number.

1. (25 points). Derive the MLE for μ (denoted $\widehat{\mu}_n$) and an approximate 95% confidence interval for $g(\mu)$ based on $\widehat{\mu}_n$.
2. (25 points). Show that the UMVUE for $g(\mu)$ is

$$\widetilde{g}(\mu) = \Phi\left(\frac{c - \overline{X}_n}{\sqrt{1 - n^{-1}}}\right).$$

   To derive this, you may use the following results:

   - $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for μ;
   - $X_1 - \overline{X}_n$ and $\overline{X}_n$ are independent. This follows directly from Basu's Theorem because $\overline{X}_n$ is complete and sufficient for μ; $X_1 - \overline{X}_n$ is ancillary (i.e., its distribution does not depend on μ) and hence the independence result follows.

THEORY, Problem 6

Consider the one-way analysis of variance (ANOVA) model:

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}, i = 1, 2, 3 \text{ and } j = 1, \cdots, n,$$

where $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$.

(a) [10 pt] Provide the design matrix X and argue that the rank of X is less than the number of its columns.

(b) [20 pt] Give an example of an estimable function and an example of non-estimable function. Be sure to justify.

From (a) we know that least squares estimates (LSE) of the parameters are not unique; in other words, the parameters are not identifiable. One way to obtain a unique LSE is to add an identifiability constraint. From now on, we use the sum-to-zero constraint: $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

(c) [20 pt] Show that $\hat{\mu} = \bar{Y}_{\cdot\cdot}, \hat{\alpha}_i = \bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}$ is the LSE under this constraint. Hint: Show that under the sum-to-zero constraint, we have

$$\sum_{i,j}(Y_{ij} - \mu - \alpha_i)^2 = \sum_{i,j}(Y_{ij} - \bar{Y}_{i\cdot})^2 + \sum_{i,j}(\bar{Y}_{\cdot\cdot} - \mu)^2 + \sum_{i,j}(\alpha_i - \bar{Y}_{i\cdot} + \bar{Y}_{\cdot\cdot})^2$$

(d) [10 pt] Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$. What distribution does $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)^T$ follow? Justify.

(e) [20 pt] Use the results you derived in (d) and other things your learned from STAT200C to show that

$$\frac{SSA/2}{SSE/(3n-3)} \overset{H_0}{\sim} F_{2,3n-3},$$

where $SSA = n\hat{\alpha}^T\hat{\alpha} = n\sum_{i=1}^{3}(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot})^2$, $SSE = \sum_{i=1}^{3}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{i\cdot})^2$, and $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = 0$.

(f) [20 pt] Consider another null hypothesis: $H_0^* : \alpha_1 = \alpha_2$. Derive a test. Be sure to justify its null distribution, i.e., the distribution of your test statistic when the null hypothesis is true.

THEORY, Problem 7

Let Y be a random vector that follows a multivariate normal distribution. We partition the vector into two vectors. Accordingly, we also partition the mean vector and the variance covariance matrix:

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix})$$

We assume that the variance covariance matrix of Y is positive definite.

(a) [10 pt] Justify that

$$[Y_1 - \Sigma_{12}\Sigma_{22}^{-1}Y_2] \sim N(\mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2, \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

(b) [10 pt] Show that $[Y_1 - \Sigma_{12}\Sigma_{22}^{-1}Y_2]$ and Y_2 are independent.

(c) [20 pt] Use the results in (a) and (b) to argue that

$$[Y_1 | Y_2 = y_2] \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}).$$

For the following questions, suppose $X_1, X_2, X_3 \overset{iid}{\sim} N(\mu, 1)$. Let $\bar{X} = (X_1 + X_2 + X_3)/3$.

(d) [20 pt] Show that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \bar{X} \end{pmatrix} \sim N(\begin{pmatrix} \mu \\ \mu \\ \mu \\ \mu \end{pmatrix}, \begin{pmatrix} I_3 & \mathbf{1}_3/3 \\ \mathbf{1}_3^T/3 & 1/3 \end{pmatrix})$$

where I_3 is the 3×3 identity matrix and $\mathbf{1}_3$ is the vector consisting of three 1's.

(e) [20 pt] Use the results in (c) and (d) to show that

$$[\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} | \bar{X} = 0] \sim N(0, I_3 - \frac{1}{3}\mathbf{1}_3\mathbf{1}_3^T).$$

(f) [20 pt] Use the result in (e) to prove that

$$[(X_1^2 + X_2^2 + X_3^2) | \bar{X} = 0] \sim \chi^2_{df}.$$

Be sure to derive the value of df.

Table 1: Common distributions and densities.

Distribution	Notation	Density
Bernoulli	$\mathrm{Bern}(\theta)$	$f(y\|\theta)=\theta^y(1-\theta)^{1-y}$
Binomial	$\mathrm{Bin}(n,\theta)$	$f(y\|\theta)=\binom{n}{y}\theta^y(1-\theta)^{n-y}$
Multinomial	$\mathrm{Multi}(n;\theta_1,\theta_2,\ldots,\theta_K)$	$f(y\|\theta)=\frac{n!}{y_1!y_2!\ldots y_K!}\theta_1^{y_1}\theta_2^{y_2}\cdots\theta_K^{y_K}$
Beta	$\mathrm{Beta}(a,b)$	$p(\theta)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}I_{(0,1)}(\theta)$
Uniform	$U(a,b)$	$p(\theta)=\frac{I_{(a,b)}(\theta)}{b-a}$
Poisson	$\mathrm{Pois}(\theta)$	$f(y\|\theta)=\theta^y e^{-\theta}/y!$
Exponential	$\mathrm{Exp}(\theta)$	$f(y\|\theta)=\theta e^{-\theta y}I_{(0,\infty)}(y)$
Gamma	$\mathrm{Gamma}(a,b)$	$p(\theta)=[b^a/\Gamma(a)]\theta^{a-1}e^{-b\theta}I_{(0,\infty)}(\theta)$
Chi-squared	$\chi^2(n)$	Same as $\mathrm{Gamma}(n/2,1/2)$
Weibull	$\mathrm{Weib}(\alpha,\theta)$	$f(y\|\theta)=\theta\alpha y^{\alpha-1}\exp\left(-\theta y^\alpha\right)I_{(0,\infty)}(\theta)$
Normal	$N(\theta,1/\tau)$	$f(y\|\theta,\tau)=(\sqrt{\tau/2\pi})\exp\left[-\tau(y-\theta)^2/2\right]$
Student's t	$t(n,\theta,\sigma)$	$f(y\|\theta)=[1+(y-\theta)^2/n\sigma^2]^{(n+1)/2}$
		$\times\,\Gamma[(n+1)/2]/\Gamma(n/2)\sigma\sqrt{n\pi}$
Cauchy	$\mathrm{Cauchy}(\theta)$	same as $t(1,\theta,1)$
Dirichlet	$\mathrm{Dirichlet}(a_1,a_2,a_3)$	$p(\theta)=\Gamma(a_1+a_2+a_3)/\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)$
		$\times\,\theta_1^{a_1-1}\theta_2^{a_2-1}(1-\theta_1-\theta_2)^{a_3-1}$
		$\times\,I_{(0,1)}(\theta_1)I_{(0,1)}(\theta_2)I_{(0,1)}(1-\theta_1-\theta_2)$

Table 2: Means, Modes, and Variances.

Distribution	Mean	Mode	Variance
Bern(θ)	θ	0 if $\theta < .5$ 1 if $\theta > .5$	$\theta(1-\theta)$
Bin(n, θ)	$n\theta$	integer closest to $n\theta$	$n\theta(1-\theta)$
Beta(a, b)	$a/(a+b)$	$(a-1)/(a+b-2)$ if $a > 1, b \geq 1$	$ab/(a+b)^2(a+b+1)$
$U(a, b)$	$.5(a+b)$	everything a to b	$(b-a)^2/12$
Pois(θ)	θ	integer closest to θ	θ
Exp(θ)	$1/\theta$	0	$1/\theta^2$
Gamma(a, b)	a/b	$(a-1)/b$ if $a > 1$	a/b^2
$\chi^2(n)$	n	$n-2$ if $n > 2$	$2n$
Weib(α, θ)	$\Gamma[(\alpha+1)/\alpha]/\theta$	$[(\alpha-1)/\alpha]^{1/\alpha}/\theta$	$\Gamma[(\alpha+2)/\alpha] - \mu^2$
$N(\theta, 1/\tau)$	θ	θ	$1/\tau$
$t(n, \theta, \sigma)$	θ if $n \geq 2$	θ	$\sigma^2 n/(n-2)$ if $n \geq 3$
Cauchy(θ)	Undefined	θ	Undefined