Second Year Exam
2021

This is a closed book and notes examination. You are to answer exactly 5 of the following 6 questions. Use your time wisely. Clearly justify each step. The 5 questions you choose to answer will be worth equal credit. Please write only on one side of each page.

1. Let $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$ be two σ-finite measure spaces.

   (a) Explain what σ-finite means, and give an example of a measure space that is not σ-finite.

   (b) Let $\mathcal{G} \equiv \{A \times B : \ A \in \mathcal{F}_1, B \in \mathcal{F}_2\}$. Show that if $S \in \mathcal{G}$, then S^c can be expressed as a finite union of disjoint sets in $\mathcal{G}$.

   (c) Describe what type of sets are included in the smallest field (not σ-field) that contains $\mathcal{G}$. Justify.

   (d) Define μ_3 on $\mathcal{G}$ (as in part(b)) by $\mu_3(A \times B) = \mu_1(A)\mu_2(B)$ for $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Show that if $A \times B = \cup_{j=1}^{\infty}(A_j \times B_j)$ where $A, A_j \in \mathcal{F}_1$ and $B, B_j \in \mathcal{F}_2$, and $A_j \times B_j, j = 1, 2, \ldots$ are disjoint, then $\mu_3(A \times B) = \sum_{j=1}^{\infty} \mu_3(A_j \times B_j)$.

   (e) Let $\mathcal{F}_3$ be the σ-field generated by $\mathcal{G}$ (as in part (b)). Which theorem guarantees that μ_3 can be extended to $\mathcal{F}_3$? Explain.

2. Let X be a nonnegative random variable with distribution function $F(x)$.

   (a) Show that if $EX < \infty$ then $x(1-F(x)) \to 0$ as $x \to \infty$. (Hint: express $x(1-F(x))$ as the expected value of some random variable and justify taking the limit inside the expectation.)

   (b) Let $\epsilon > 0$. Show that

   $$\int_1^{\infty} (1 - F(\epsilon x))dx \le \sum_{n=1}^{\infty} \Pr(X > n\epsilon) \le \int_0^{\infty} (1 - F(\epsilon x))dx.$$

   (c) Use Fubini's Theorem to show that $\int_0^{\infty}(1 - F(\epsilon x))dx = EX/\epsilon$.

   (d) Suppose $X_1, X_2, \ldots,$ are iid with distribution function F. Use parts (b) and (c) to show that if $EX < \infty$ then $X_n/n \to 0$ almost surely as $n \to \infty$.

3. Let $X_n \sim \text{Poisson}(\lambda)$ independently, $n = 1, 2, \ldots$. Recall the Poisson PMF $\Pr(X_n = k) = \lambda^k e^{-\lambda}/k!$ for $k = 0, 1, \ldots$, where $\lambda > 0$. Define $S_n = \sum_{j=1}^{n} I(X_j = 0)$ and $T_n = \sum_{j=1}^{n} X_j$, where $I(\cdot)$ denotes the indicator function.

   (a) Derive the characteristic function of S_n.

   (b) Show that S_n/T_n tends to a constant in probability as $n \to \infty$. Identify this constant.

   (c) Define $W_n = \sum_{j=1}^{n} jX_j$. Identify suitable constants $b_n > 0$ such that $W_n/b_n \to 1$ in probability. Justify.

   (d) Define W_n as in part (c). Identify constants a_n, c_n such that $(W_n - a_n)/c_n \xrightarrow{D} \text{N}(0, 1)$. Justify.

4. (a) Consider a random variable X defined as $X(\omega) = \omega$ for every $\omega \in [0, 1]$ equipped with Borel sigma field and Lebesgue measure. Let $Y = I(X < 1/3) + 2I(1/3 \le X < 2/3) + 3I(X \ge 2/3)$, and $Z = X^2$. What is the sigma field generated by Y? What is the sigma field generated by Z?

   (b) Following part (a), find $\text{E}(X \mid Y)$ and $\text{E}(X \mid Z)$. Explain.

   (c) Now consider a general situation. Suppose there is a random variable X defined on $(\Omega, \mathcal{F})$. Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two sub sigma-fields of $\mathcal{F}$ and $\mathcal{F}_1 \neq \mathcal{F}_2$. Is it true that $\text{E}(\text{E}(X \mid \mathcal{F}_1) \mid \mathcal{F}_2) = \text{E}(\text{E}(X \mid \mathcal{F}_2) \mid \mathcal{F}_1)$? Explain or prove your answer.

(d) Following (c), compare those two terms, $\mathrm{E}(X - \mathrm{E}(X \mid \mathcal{F}_1))^2$ and $\mathrm{E}(X - \mathrm{E}(X \mid \mathcal{F}_2))^2$, which one is larger? Explain or prove your answer.

5. Let $\theta > 0$ be a parameter of interest and let X follow a uniform distribution on $(0, \theta]$. For the entire problem, consider a loss function $L(\theta, \hat{\theta}) = |\theta - \hat{\theta}|/\theta$ for an estimator $\hat{\theta}$.

   (a) Consider an estimator $T(X) = cX$ for some constant $c \geq 1$. Find the risk of $T(X)$.

   (b) Now find the value of c such that the risk of $T(X)$ you obtained in part (a) is minimized. Denote the corresponding estimator by $\hat{\theta}$.

   (c) Now the goal is to show that $\hat{\theta}$ is an extended Bayes estimator (limit of a sequence of Bayes estimators). Consider a Pareto distribution with parameter $x_m > 0, \alpha > 0$. Its pdf, expectation, and median are

$$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}} I(x \geq x_m), \mathrm{E}(X) = \frac{\alpha x_m}{\alpha - 1} \text{ for } \alpha > 1, \mathrm{median}(X) = x_m 2^{1/\alpha}.$$

   Now show that $\hat{\theta}$ is an extended Bayes estimator under some Pareto priors.

   (d) Prove this result in general: if a Bayes rule has a constant risk (with respect to different parameter values), it must be a minimax rule.

   (e) Recall from our class: an extended Bayes rule that has a constant risk is also minimax (under some conditions). Use this result (you don't have to prove it) to show that $\hat{\theta}$ you obtained in part (b) is minimax.

6. Consider $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}[0, \theta]$. We know MLE is $X_{(n)} = \max\{X_1, \ldots, X_n\}$ in this case. Also define $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $\bar{X}_n = \sum_{i=1}^n X_i/n$.

   (a) Find a sequence of numbers a_n such that $a_n(\theta - X_{(n)})$ converges in distribution to some non-degenerating limit as $n \to \infty$.

   (b) Let $T = \frac{n+1}{n} X_{(n)}$. Find a sequence of numbers b_n such that $b_n(\theta - T)$ converges in distribution to some non-degenerating limit.

   (c) Find the variance of T and show it does not satisfy the Cramér-Rao lower bound, i.e., $\left\{ n E_\theta (\frac{\partial l(X;\theta)}{\partial \theta})^2 \right\}^{-1}$, where $l(X; \theta)$ is the log-likelihood function based on a single observation X.

   (d) Explain why the Cramér-Rao lower bound does not hold in part (c).

   (e) Compare $X_{(n)}$, $\frac{n+1}{n} X_{(n)}$, $2\bar{X}_n$, and $X_{(n)} + X_{(1)}$ as estimators for θ, which one is the best? Explain.