2018 First Year Exam
Theory Statistics 200ABC
June 22, 2018
9:00 to 1:00

• There are 7 questions on the examination. Select any 5 of them to solve. If you attempt to solve more than 5 questions, you are only to turn in the 5 you want graded. If you turn in partial solutions to more than 5 questions, only 5 will be graded.

• Each of the 5 problems you attempt to solve will be worth equal credit, with each accounting for 20% of your final score on this examination.

• Your solutions to each problem should be written on separate sheets of paper. Label each sheet with your student identification number, the problem number, and the page number of that solution written in the upper right hand corner. For example, the labeling on a page may be:

ID# 912346378
Problem 2, page 3

• You have 4 hours to complete your solution. Please be prepared to turn in your exam at 1:00 pm.

THEORY (2018), Problem 1

Geometric distribution. The following facts about geometric distribution might be useful. Let Y be a random variable that follows the geometric distiibution with success probability p. Then (i) $Pr(Y = k) = (1-p)^{k-1}p, k = 1, 2, \cdots$; $(ii) E(Y) = \frac{1}{p}$; (iii) $Var(Y) = \frac{1-p}{p^2}$; (iv) the moment generating function of Y is $M_Y(t) = \frac{pe^t}{1-(1-p)e^t}$.

Let X be a random variable that follows the exponential distribution with rate parameter 1.

(a) Describe a point process that connects X to a Poisson distributed random variable.

(b) X can be connected to another discrete random variable by the ceiling function $C(x)$, which maps x to the least integer that is greater than or equal to x. For example, $C(1.0) = 1.0, C(1.01) = 2, C(1.99) = 2$. Let

$$Y_a = C(X/a),$$

where a is a positive number. Show that Y_a follows the geometric distribution with success probability $p = 1 - e^{-a}$.

(c) Show that aY_a converges in distribution to X as $a \to 0$.

(d) Let $U = e^{-X}$. Show that $U \sim U(0, 1)$, i.e, the uniform distribution between 0 and 1.

(e) Prove that the conditional distribution of U given $Y_a = k$ is $U(e^{-ak}, e^{-a(k-1)})$.

THEORY (2018), Problem 2

Suppose $(X_1, \cdots, X_n)$ is a random sample from the standard Normal distribution. Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, i.e., the sample mean. Answer the following questions. Be sure to justify your answer.

(a) As $n \to \infty$, to which constant does $(\bar{X}_n)^2$ converge in probability?

(b) For a fixed n, what is the distribution of $\frac{1}{n}\sum_{i=1}^n X_i^2$?

(c) As $n \to \infty$, to which constant does $\frac{1}{n}\sum_{i=1}^n X_i^2$ converge to in probability?

(d) Consider $\sqrt{n}(\frac{1}{n}\sum_{i=1}^n X_i^2 - 1)$. What is its limiting distribution?

(e) Find the asymptotic distribution of $\sqrt{n}(\sqrt{\frac{1}{n}\sum_{i=1}^n X_i^2} - 1)$.

THEORY (2018), Problem 3

Let $X_1, \ldots, X_n$ be i.i.d. from $f_X(x|\theta) = \theta x^{-(\theta+1)}$, $x \geq 1$, $\theta > 0$.

(a) Find the method of moment estimator for θ based on the first moment of X, assuming $\theta > 1$.

(b) Derive the asymptotic distribution of the estimator given in (a).

(c) Find the maximum likelihood estimator for θ.

(d) Provide the asymptotic distribution of the estimator given in (c)

(e) Find the maximum likelihood estimator for $1/\theta$.

(f) Is the estimator given in (e) an unbiased estimator for $1/\theta$? Provide your argument.

(g) Provide the asymptotic distribution of the estimator given in (e).

END OF QUESTION (3)

THEORY (2018), Problem 4

Consider a random sample $X_1, X_2, \ldots, X_n$ from $N(\mu, \sigma^2)$, where σ^2 is known. Let $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$, which is the maximum likelihood estimator for μ.

(a) Show that $\bar{X}$ is a sufficient statistic for μ.

(b) Is $\bar{X}$ minimal sufficient? Provide detailed argument to support your answer.

(c) Is $\bar{X}$ a complete statistic? Provide detailed argument to support your answer.

(d) Show that the size α likelihood ratio test for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ is the z-test that rejects H_0 when

$$\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \geq z_\alpha.$$

(e) Is the test in (d) an unbiased test? Provide detailed argument to support your answer.

(f) Is the test in (d) a uniformly most powerful test? Provide detailed argument to support your answer.

END OF QUESTION (4)

THEORY (2018), Problem 5

Let X_{org} be an $n \times p$ design matrix with rank p. Consider the following linear model

$$Y = X_{org}\beta_{org} + \epsilon \text{ (model 1)}$$

(a) Let $X = X_{org}(X_{org}^T X_{org})^{-1/2}$. Note that the columns of X have been decorrelated and standardized, i.e., $X^T X = I_p$. If we use X as the design matrix and fit the following model

$$Y = X\beta + \epsilon \text{ (model 2)}$$

Show that model (1) and model (2) are reparameterization to each other by verifying that the two models lead to the same fitted value for Y.

(b) Let A be a $q \times p$ known matrix with rank q. Consider the third model

$$Y = X(I - A^T(AA^T)^{-1}A)\gamma + \epsilon \text{ (model 3)}$$

Is model (3) a reparameterization of model (2)? Is the coefficient vector γ estimable? Justify your answer.

(c) Verify that an least squares estimate (LSE) of γ is $\hat{\gamma} = (I - A^T(AA^T)^{-1}A)\hat{\beta}$ by showing that $\tilde{X}^T\tilde{X}\hat{\gamma} = \tilde{X}^T Y$, where $\tilde{X} = X(I - A^T(AA^T)^{-1}A)$ and $\hat{\beta} = (X^T X)^{-1}X^T Y = X^T Y$. Hint: This is an easy problem. Just use the fact that $X^T X = I_p$.

(d) One way to compare two linear models is to compare their residual sum of squares (RSS). Let RSS_i denote the RSS of model i, $i = 2, 3$. Show that

$$RSS_3 - RSS_2 = \hat{\beta}^T A^T(AA^T)^{-1}A\hat{\beta} = Y^T X A^T(AA^T)^{-1}AX^T Y$$

Hint: $RSS_3 = ||Y - \tilde{X}\hat{\gamma}|| = \cdots$

(e) Let's assume that model (3) is the true model and

$$Y \sim N(X(I - A^T(AA^T)^{-1}A)\gamma, \sigma^2 I_n)$$

Prove that

$$\frac{(RSS_3 - RSS_2)/q}{RSS_2/(n-p)} \sim F_{q,n-p}$$

THEORY (2018), Problem 6

Consider the one-way random effects model

$$Y_{ij} = \mu + a_i + \epsilon_{ij}, i = 1, \cdots, I; j = 1, \cdots, J$$

with $a_i \overset{iid}{\sim} N(0, \sigma_A^2)$ and $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$. Assume that all the random elements are independent. Define the following sums of squares:

$$SSA = J \sum_{i=1}^{I} (\bar{Y}_{i.} - \bar{Y}_{..})^2, SSE = \sum_{i=1}^{I} \sum_{j=1}^{J} (\bar{Y}_{ij} - \bar{Y}_{i.})^2$$

(a) Show that $SSA = \sum_{i=1}^{I} (X_i - \bar{X})^2$ where $X_i = \sqrt{J}(a_i + \bar{\epsilon}_{i.})$, $\bar{X} = \sum_{i=1}^{I} X_i / I$, and $\bar{\epsilon}_{i.} = \sum_{j=1}^{J} \epsilon_{ij} / J$.

(b) Find the distribution of X_i and use the result to show that $\frac{SSA}{J\sigma_A^2 + \sigma^2} \sim \chi^2_{I-1}$.

(c) Show that $SSE = \sum_{i=1}^{I} \sum_{j=1}^{J} (\epsilon_{ij} - \bar{\epsilon}_{i.})^2$ and then show that $SSE/\sigma^2 \sim \chi^2_{I(J-1)}$.

(d) Are SSA and SSE independent? Justify your answer.

(e) Suppose we are interested in the signal-to-noise ratio $\rho = \sigma_A^2/\sigma^2$. Construct a $100(1 - \alpha)\%$ confidence interval for ρ.

THEORY (2018), Problem 7

Suppose $X_1, \ldots, X_n$ are i.i.d. samples from a geometric distribution with $f(x|\theta) = \theta(1-\theta)^x$ for $x = 0, 1, 2, \ldots$ and $0 \leq \theta \leq 1$.

(a) Find the MLE of θ. Is it unbiased?

(b) Use the Central Limit Theorem, and the Delta method to find the limiting distribution of the MLE.

(c) Show that this distribution is the same as the distribution according the theorem on asymptotic normality of MLE.

(d) Set $\eta = 1/\theta$. Find the MLE for η. Is it unbiased?

(e) Find the Fisher information for θ and use it to find the Fisher information for η.

(f) Does the MLE of η attains the Cramer-Rao lower bound?

Table 1: Common distributions and densities.

Distribution	Notation	Density
Bernoulli	Bern(θ)	$f(y\|\theta) = \theta^y(1-\theta)^{1-y}$
Binomial	Bin(n, θ)	$f(y\|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$
Multinomial	Multi($n; \theta_1, \theta_2, \ldots, \theta_K$)	$f(y\|\theta) = \frac{n!}{y_1!y_2!\ldots y_K!}\theta_1^{y_1}\theta_2^{y_2}\cdots\theta_K^{y_K}$
Beta	Beta(a, b)	$p(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}I_{(0,1)}(\theta)$
Uniform	$U(a, b)$	$p(\theta) = \frac{I_{(a,b)}(\theta)}{b-a}$
Poisson	Pois(θ)	$f(y\|\theta) = \theta^y e^{-\theta}/y!$
Exponential	Exp(θ)	$f(y\|\theta) = \theta e^{-\theta y}I_{(0,\infty)}(y)$
Gamma	Gamma(a, b)	$p(\theta) = [b^a/\Gamma(a)]\theta^{a-1}e^{-b\theta}I_{(0,\infty)}(\theta)$
Chi-squared	$\chi^2(n)$	Same as Gamma($n/2, 1/2$)
Weibull	Weib(α, θ)	$f(y\|\theta) = \theta\alpha y^{\alpha-1}\exp\left(-\theta y^\alpha\right)I_{(0,\infty)}(\theta)$
Normal	$N(\theta, 1/\tau)$	$f(y\|\theta, \tau) = (\sqrt{\tau/2\pi})\exp\left[-\tau(y-\theta)^2/2\right]$
Student's t	$t(n, \theta, \sigma)$	$f(y\|\theta) = [1+(y-\theta)^2/n\sigma^2]^{(n+1)/2}$
		$\times\,\Gamma[(n+1)/2]/\Gamma(n/2)\sigma\sqrt{n\pi}$
Cauchy	Cauchy(θ)	same as $t(1, \theta, 1)$
Dirichlet	Dirichlet(a_1, a_2, a_3)	$p(\theta) = \Gamma(a_1+a_2+a_3)/\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)$
		$\times\,\theta_1^{a_1-1}\theta_2^{a_2-1}(1-\theta_1-\theta_2)^{a_3-1}$
		$\times\,I_{(0,1)}(\theta_1)I_{(0,1)}(\theta_2)I_{(0,1)}(1-\theta_1-\theta_2)$

Table 2: Means, Modes, and Variances.

Distribution	Mean	Mode	Variance
Bern(θ)	θ	0 if $\theta < .5$ 1 if $\theta > .5$	$\theta(1-\theta)$
Bin(n, θ)	$n\theta$	integer closest to $n\theta$	$n\theta(1-\theta)$
Beta(a, b)	$a/(a+b)$	$(a-1)/(a+b-2)$ if $a > 1, b \geq 1$	$ab/(a+b)^2(a+b+1)$
$U(a, b)$	$.5(a+b)$	everything a to b	$(b-a)^2/12$
Pois(θ)	θ	integer closest to θ	θ
Exp(θ)	$1/\theta$	0	$1/\theta^2$
Gamma(a, b)	a/b	$(a-1)/b$ if $a > 1$	a/b^2
$\chi^2(n)$	n	$n-2$ if $n > 2$	$2n$
Weib(α, θ)	$\Gamma[(\alpha+1)/\alpha]/\theta$	$[(\alpha-1)/\alpha]^{1/\alpha}/\theta$	$\Gamma[(\alpha+2)/\alpha] - \mu^2$
$N(\theta, 1/\tau)$	θ	θ	$1/\tau$
$t(n, \theta, \sigma)$	θ if $n \geq 2$	θ	$\sigma^2 n/(n-2)$ if $n \geq 3$
Cauchy(θ)	Undefined	θ	Undefined